Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel
International Advanced Level

Centre Number	Candidate Number

Thursday 7 January 2021

Morning (Time: 1 hour 30 minutes)	Paper Reference **WMA14/01**

Mathematics

International Advanced Subsidiary/Advanced Level
Pure Mathematics P4

You must have: Mathematical Formulae and Statistical Tables (Lilac), calculator	Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 10 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over ▶

Pearson

Leave blank

1. (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of

$$\left(\frac{1}{4} - 5x\right)^{\frac{1}{2}} \qquad |x| < \frac{1}{20}$$

giving each coefficient in its simplest form.

(5)

By substituting $x = \frac{1}{100}$ into the answer for (a),

(b) find an approximation for $\sqrt{5}$

Give your answer in the form $\frac{a}{b}$ where a and b are integers to be found.

(2)

(a) Get the formula for the Binomial Expansion from the formula booklet:

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$, Range of validity $|x|<1$

We see that the given expansion is in form $(a+bx)^n$, we need to manipulate to get $(1+ax)^n$.

$$\left(\frac{1}{4} - 5x\right)^{\frac{1}{2}} = \left(\frac{1}{4}\right)^{\frac{1}{2}}(1-20x)^{\frac{1}{2}}$$

$$= \frac{1}{2}(1-20x)^{\frac{1}{2}} \quad \text{B1}$$

Substitute into the Formula:

$$\frac{1}{2}(1-20x)^{\frac{1}{2}} = \frac{1}{2}\left(1 + \left(\tfrac{1}{2}\right)(-20x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-20x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-20x)^3 + \ldots\right) \quad \text{M1 A1}$$

$$= \frac{1}{2}\left(1 - 10x + \left(-\tfrac{1}{8}\right)(400x^2) + \left(\tfrac{1}{16}\right)(-8000x^3) + \ldots\right)$$

$$= \frac{1}{2}(1 - 10x - 50x^2 - 500x^3 + \ldots)$$

$$= \frac{1}{2} - 5x - 25x^2 - 250x^3 + \ldots \quad \text{A1 A1}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Leave blank

DO NOT WRITE IN THIS AREA

Question 1 continued

(b) Substitute $x = \frac{1}{100}$ into LHS of expansion:

$\left(\frac{1}{4} - \frac{5}{100}\right)^{\frac{1}{2}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$.

Hence we need to multiply our expansion by 5 to get rid of the $\frac{1}{5}$, to get the needed $\sqrt{5}$.

$\sqrt{5} \approx 5 \times \left(\frac{1}{2} - 5\left(\frac{1}{100}\right) - 25\left(\frac{1}{100}\right)^2 - 250\left(\frac{1}{100}\right)^3 + \ldots\right)$ M1

$\approx 5 \times \frac{1789}{4000}$

$\therefore \sqrt{5} \approx \frac{1789}{800}$ A1

Q1

(Total 7 marks)

Leave blank

2.

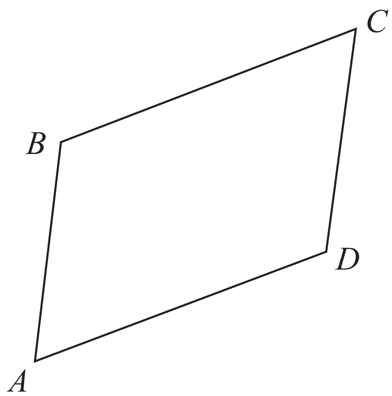


Figure 1

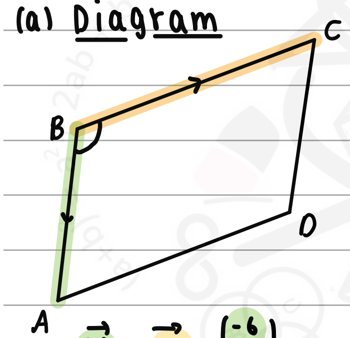
Figure 1 shows a sketch of parallelogram $ABCD$.

Given that $\overrightarrow{AB} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{BC} = 2\mathbf{i} + 5\mathbf{j} + 8\mathbf{k}$

(a) find the size of angle ABC, giving your answer in degrees, to 2 decimal places. **(3)**

(b) Find the area of parallelogram $ABCD$, giving your answer to one decimal place. **(2)**

(a) Diagram



$\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix}-6\\2\\-3\end{pmatrix}$

To get the angle ABC use formula:

$\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$ Formula for dot product: $\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}d\\e\\f\end{pmatrix} = ad + be + cf$

Formula for Magnitude of a vector:

let $\overrightarrow{AB} = \begin{pmatrix}a\\b\\c\end{pmatrix}$.

$|\overrightarrow{AB}| = \sqrt{a^2 + b^2 + c^2}$

Substitute in $\overrightarrow{BA}$ and $\overrightarrow{BC}$: (Why not $\overrightarrow{AB}$? Because the formula needs the vectors to be going "out" of the angle.)

$$\cos\theta = \frac{\begin{pmatrix}-6\\2\\-3\end{pmatrix}\cdot\begin{pmatrix}2\\5\\8\end{pmatrix}}{\sqrt{6^2+2^2+3^2}\sqrt{2^2+5^2+8^2}}$$

$$= \frac{-12+10-24}{\sqrt{49}\sqrt{93}}$$ dM1

$$\cos\theta = \frac{-26}{\sqrt{49}\sqrt{93}}$$ M1

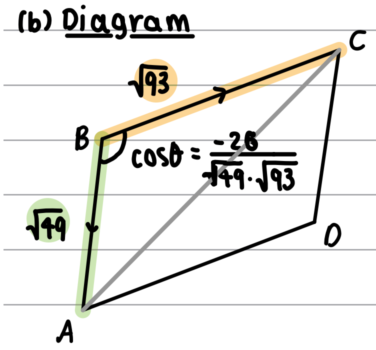
$\theta = 112.65°$ A1

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Leave blank

Question 2 continued

(b) Diagram



We got the magnitudes of the sides and the angle between them (ABC) in (a).

Hence find the area of ΔABC and double that to get the area of the parallelogram

Use Formula:

$\text{area} = \frac{1}{2}ab\sin C$

Get $\sin\theta$:

$\sin^2\theta + \cos^2\theta = 1$ use Identity

$\therefore \sin\theta = \sqrt{1-\cos^2\theta}$ Rearrange

Substitute into formula:

$\text{area} = \frac{1}{2} \times \sqrt{93} \times \sqrt{49} \times \sqrt{1-\left(\frac{-26}{\sqrt{93}\sqrt{49}}\right)^2}$ M1

$= 31.149$

$\downarrow \times 2$

area of parallelogram $= 62.3$ A1

Q2

(Total 5 marks)

Turn over

P 6 7 7 5 4 A 0 5 2 8

Leave blank

3. Prove by contradiction that there is no greatest odd integer.

(2)

Make an Assumption: we want our assumption to be the opposite of what we are trying to prove.

"Assume there is a largest odd integer"

M1

In the Proof, we are trying to contradict the assumption

Let the largest odd integer to be $2k+1$.

$\therefore$ There also exists $2k+3$.

Hence $2k+3$ is an odd number larger than $2k+1$, which is a contradiction to our assumption, hence proven by contradiction that there is no largest odd integer.

A1

Leave blank

Question 3 continued

Q3

(Total 2 marks)

Leave blank

4. The curve C is defined by the parametric equations

$$x = \frac{1}{t} + 2 \qquad y = \frac{1-2t}{3+t} \qquad t > 0$$

(a) Show that the equation of C can be written in the form $y = \mathrm{g}(x)$ where g is the function

$$\mathrm{g}(x) = \frac{ax+b}{cx+d} \qquad x > k$$

where a, b, c, d and k are integers to be found.

(5)

(b) Hence, or otherwise, state the range of g.

(2)

(a) Make t the subject of $x = \ldots$:

$$x = \frac{1}{t} + 2$$

$$x - 2 = \frac{1}{t}$$

$$t = \frac{1}{x-2}$$

we are told that $t > 0$. For this to be met, $x > 2$ must be true $\therefore k = 2$

B1

Substitute our found t in terms of x into $y = \ldots$ and simplify:

$$y = \frac{1 - 2\left(\frac{1}{x-2}\right)}{3 + \frac{1}{x-2}}$$

M1A1

$$= \frac{\frac{(x-2)}{(x-2)} - \frac{2}{(x-2)}}{\frac{3(x-2)}{(x-2)} + \frac{1}{(x-2)}}$$

Make common denominators both up and down

$$= \frac{\frac{x-4}{x-2}}{\frac{3x-5}{x-2}}$$

$$y = \frac{x-4}{3x-5}, \quad x > 2$$

A1A1

(b) Use domain $x > 2$ (found) to get lower bound:

$$y = \frac{2-4}{3(2)-5}$$

$$y = \frac{-2}{1} = -2 \quad \text{lower bound}$$

Use domain $t > 0$ (given) to get upper bound: M1

$$y = \frac{1-2(0)}{3+0}$$

$$\therefore \; y = \frac{1}{3} \quad \text{upper bound}$$

Hence Range: $-2 < g(x) < \frac{1}{3}$ A1

P 6 7 7 5 4 A 0 8 2 8

Leave blank

Question 4 continued

Q4

(Total 7 marks)

Leave blank

5. **In this question you should show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Using the substitution $u = 3 + \sqrt{2x - 1}$ find the exact value of

$$\int_1^{13} \frac{4}{3 + \sqrt{2x - 1}}\, dx$$

giving your answer in the form $p + q \ln 2$, where p and q are integers to be found.

(8)

(a) To use the given substitution $u = 3 + \sqrt{2x-1}$, we need to find new limits and $dx = \ldots$

New Limits:

x		u
1	$\longrightarrow 3+\sqrt{2(1)-1} \longrightarrow$	4
13	$\longrightarrow 3+\sqrt{2(13)-1} \longrightarrow$	8

To get dx, differentiate $u = 3 + \sqrt{2x-1}$:

$u = 3 + (2x-1)^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}(2x-1)^{-\frac{1}{2}} \times 2$

$dx = (2x-1)^{\frac{1}{2}} du$

M1A1

Substitute into the Integration:

$$\int_1^{13} \frac{4}{3+\sqrt{2x-1}}\, dx$$

u

$= \int_4^8 \frac{4}{u} \times \sqrt{2x-1}\, du$ $\quad u - 3 = \sqrt{2x-1}$

$= \int_4^8 \frac{4}{u}(u-3)\, du$ M1

$= \int_4^8 \frac{4u-12}{u}\, du$

$= \int_4^8 4 - \frac{12}{u}\, du$ Split fraction

dM1

$= \left[4u - 12\ln u\right]_4^8$ Integrate

ddM1A1

$= (4(8) - 12\ln 8) - (4(4) - 12\ln(4))$ sub. in limits

$= 32 - 12\ln 8 - 16 + 12\ln 4$ M1

$= 16 + 12\ln 4 - 12\ln 8$

$= 16 + 12\ln\frac{4}{8}$ apply ln law $\ln a - \ln b = \ln\frac{a}{b}$

$= 16 + 12\ln\frac{1}{2}$

$= 16 + 12\ln 2^{-1}$

$= 16 - 12\ln 2$ A1 apply ln law $\ln a^b = b\ln a$

P 6 7 7 5 4 A 0 1 0 2 8

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Leave blank

Question 5 continued

Q5

(Total 8 marks)

Turn over

Leave blank

6. A curve has equation

$$4y^2 + 3x = 6ye^{-2x}$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(5)

The curve crosses the y-axis at the origin and at the point P.

(b) Find the equation of the normal to the curve at P, writing your answer in the form $y = mx + c$ where m and c are constants to be found.

(4)

(a) ☆ Implicit differentiation:

When differentiating y, multiply by $\frac{dy}{dx}$

$$4y^2 + 3x = 6ye^{-2x}$$

By Parts: $\frac{d}{dx}(uv) = uv' + vu'$

$$8y\frac{dy}{dx} + 3 = 6e^{-2x}\frac{dy}{dx} - 12ye^{-2x}$$ B1M1A1

$$3 + 12ye^{-2x} = 6e^{-2x}\frac{dy}{dx} - 8y\frac{dy}{dx}$$ collect all $\frac{dy}{dx}$ terms on one side

$$3 + 12ye^{-2x} = \frac{dy}{dx}(6e^{-2x} - 8y)$$ Factor out $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{3 + 12ye^{-2x}}{6e^{-2x} - 8y}$$ M1A1

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

P 6 7 7 5 4 A 0 1 2 2 8

Leave blank

Question 6 continued

(b) Get coordinates of P using the given fact that P is on the y-axis, $\therefore x=0$.

Substitute into given equation:

$$4y^2+3(0)=6ye^{-2(0)}$$

$$4y^2=6y$$

$$4y^2-6y=0$$

$$y(4y-6)=0$$

$$y=0 \qquad y=\frac{3}{2} \quad \text{B1}$$

$\therefore P(0,1.5)$

Use found $\frac{dy}{dx}$ to get gradient of tangent at P:

$$\frac{dy}{dx}=\frac{3+12(1.5)e^{-2(0)}}{6e^{-2(0)}-8(1.5)} \quad \text{M1}$$

$$=\frac{3+18}{-6}=-\frac{7}{2} \quad m \text{ of tangent}$$

$m(\text{normal})=-\frac{1}{m(\text{tangent})}$ $\quad\therefore m(\text{normal})=\frac{2}{7}$

Use formula $y-y_1=m(x-x_1)$ to get the equation of the line:

$m=\frac{2}{7}$, $P(0,\frac{3}{2})$:

$$y-\frac{3}{2}=\frac{2}{7}(x-0) \quad \text{dM1}$$

$$y=\frac{2}{7}x+\frac{3}{2} \quad \text{A1}$$

Q6

(Total 9 marks)

P 6 7 7 5 4 A 0 1 3 2 8

Leave blank

7.

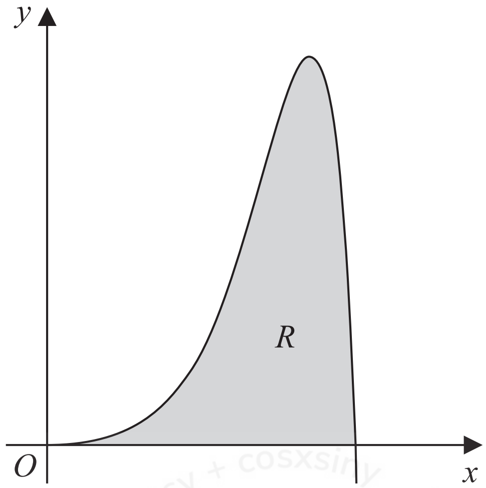


Figure 2

(a) Find $\int e^{2x} \sin x \, dx$

(5)

Figure 2 shows a sketch of part of the curve with equation

$$y = e^{2x} \sin x \qquad x \geqslant 0$$

The finite region R is bounded by the curve and the x-axis and is shown shaded in Figure 2.

(b) Show that the exact area of R is $\dfrac{e^{2\pi} + 1}{5}$

(*Solutions relying on calculator technology are not acceptable.*)

(2)

Leave blank

Question 7 continued

(a) <u>Way 1</u>

$$\int e^{2x} \sin x \, dx$$

multiplication ∴ By Parts

$\int uv' \, dx = uv - \int u'v \, dx$

$u = \sin x \xrightarrow{\frac{d}{dx}} u' = \cos x$

$v = \frac{1}{2}e^{2x} \xleftarrow{\int} v' = e^{2x}$

$$= \frac{1}{2}e^{2x}\sin x - \int \frac{1}{2}e^{2x}\cos x \, dx$$

By parts again

M1

$u = \cos x \qquad u' = -\sin x$

$v = \frac{1}{4}e^{2x} \qquad v' = \frac{1}{2}e^{2x}$

$$= \frac{1}{2}e^{2x}\sin x - \left(\frac{1}{4}e^{2x}\cos x + \int \sin x \cdot \frac{1}{4}e^{2x} \, dx\right)$$

dM1

Reintroduce our initial integral to get rid of infinite "By Parts"

$$\int e^{2x}\sin x \, dx = \frac{1}{2}e^{2x} - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x \, dx$$

A1

$$\frac{5}{4}\int e^{2x}\sin x \, dx = \frac{1}{2}e^{2x} - \frac{1}{4}e^{2x}\cos x$$

collect integral terms on LHS

ddM1

$$\int e^{2x}\sin x \, dx = \frac{2}{5}e^{2x} - \frac{1}{5}e^{2x}\cos x + c$$

A1

Leave blank

Question 7 continued

Way 2

$$\int e^{2x}\sin x\,dx$$ multiplication ∴ By Parts

$\int uv'\,dx = uv - \int u'v\,dx$

$u = e^{2x} \xrightarrow{\frac{d}{dx}} u' = 2e^{2x}$

$v = -\cos x \xleftarrow{\int} v' = \sin x$

$$= -e^{2x}\cos x - \int 2e^{2x}(-\cos x)\,dx$$ M1

$$= -e^{2x}\cos x + \int 2e^{2x}\cos x\,dx$$ By Parts again

$u = 2e^{2x} \xrightarrow{\frac{d}{dx}} u' = 4e^{2x}$

$v = \sin x \xleftarrow{\int} v' = \cos x$

$$= -e^{2x}\cos x + 2e^{2x}\sin x - \int 4e^{2x}\sin x\,dx$$ dM1

Reintroduce our initial integral to get rid of infinite "By Parts"

$$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx$$ A1

$$5\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2e^{2x}\sin x$$ collect integral terms on LHS ddM1

$$\int e^{2x}\sin x\,dx = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{x}\cos x + c$$ A1

Leave blank

Question 7 continued

(b) Lower bound is $x=0$.

Get upper bound using curve equation given:

$$0 = e^{2x}\sin x$$

$$\sin x = 0$$

$$x = 0,\ \pi$$ - upper bound

Use Integration Result we got in (a) with found limits:

$$\int_0^{\pi} e^{2x}\sin x\,dx = \left[\frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{x}\cos x\right]_0^{\pi}$$

$$= \left(\cancel{\frac{2}{5}e^{2\pi}\sin\pi}^{\,0} - \frac{1}{5}e^{2\pi}\cos\pi\right) - \left(\cancel{\frac{2}{5}e^{2(0)}\sin 0}^{\,0} - \frac{1}{5}e^{0}\cos 0\right)$$ M1

$$= \frac{1}{5}e^{2\pi} - \frac{1}{5}$$

$$= \frac{e^{2\pi}-1}{\pi}$$ A1

Q7

(Total 7 marks)

Leave blank

8. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \qquad l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -3 \\ b \end{pmatrix}$$

where λ and μ are scalar parameters and b is a constant.

Prove that for all values of $b \neq 7$, the lines l_1 and l_2 are skew.

(6)

Equate the general points of the lines:

$$\begin{pmatrix} -1+2\lambda \\ 5-\lambda \\ 4+5\lambda \end{pmatrix} = \begin{pmatrix} 2+4\mu \\ -2-3\mu \\ -5+\mu b \end{pmatrix}$$

Use x and y components to get μ and λ.

By Elimination

$$-1+2\lambda = 2+4\mu$$

$$5-\lambda = -2-3\mu \quad (\times 2) \quad 10-2\lambda = -4-6\mu \quad + \quad \text{M1}$$

$$9 = -2-2\mu \quad \text{dM1}$$

$$\mu = -\frac{11}{2} \Rightarrow -1+2\lambda = 2+4\left(-\frac{11}{2}\right)$$

$$\lambda = -\frac{19}{2} \quad \text{A1}$$

Hence we know $\mu = -\frac{11}{2}$ and $\lambda = -\frac{19}{2}$.

Substitute into z component:

$$4+5\left(-\frac{19}{2}\right) = -5+\left(-\frac{11}{2}b\right) \quad \text{ddM1}$$

$$-\frac{87}{2} = -5-\frac{11}{2}b$$

$$-\frac{77}{2} = -\frac{11}{2}b$$

$$\Rightarrow b = 7 \quad \text{A1}$$

Hence when $b=7$, the lines intersect. When $b \neq 7$, since they are not parallel and do not intersect, they are proven to be skew. A1

Leave blank

Question 8 continued

Q8

(Total 6 marks)

Leave blank

9.

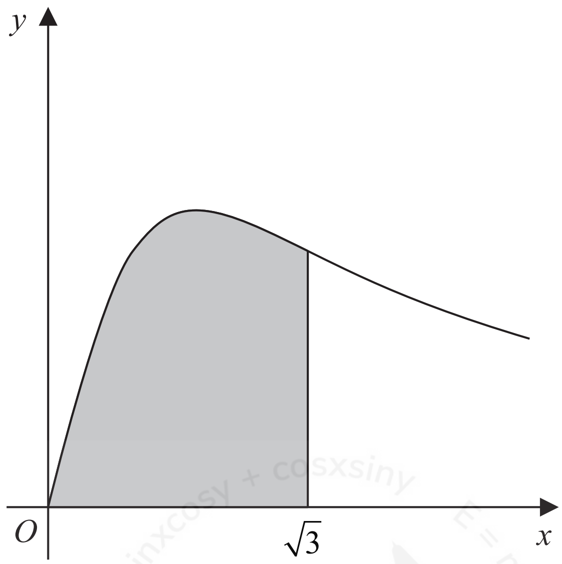


Figure 3

Figure 3 shows a sketch of part of the curve with parametric equations

$$x = \tan\theta \qquad y = 2\sin 2\theta \qquad \theta \geqslant 0$$

The finite region, shown shaded in Figure 3, is bounded by the curve, the x-axis and the line with equation $x = \sqrt{3}$

The region is rotated through 2π radians about the x-axis to form a solid of revolution.

(a) Show that the exact volume of this solid of revolution is given by

$$\int_0^k p(1 - \cos 2\theta)\, d\theta$$

where p and k are constants to be found.

(7)

(b) Hence find, by algebraic integration, the exact volume of this solid of revolution.

(3)

Leave blank

Question 9 continued

(a) Formula for Volume of Revolution around the x-axis:

$$V = \pi \int_a^b y^2 \, dx$$

★ Parametric Integration

$$\text{area} = \int_a^b y \frac{dx}{dt} \, dt$$

where a and b are the points on the x-axis bounding the area

Let's get the limits in parametric:

x	θ
$0 = \tan(\theta) \rightarrow$	0
$\sqrt{3} = \tan(\theta) \rightarrow$	$\frac{\pi}{3}$

B1

Get $\frac{dx}{d\theta}$:

$$x = \tan\theta$$

$$\frac{dx}{d\theta} = \sec^2\theta$$

Substitute into the formula:

$y = \ldots$

$$V = \pi \int_0^{\frac{\pi}{3}} (2\sin 2\theta)^2 \times \sec^2\theta \, d\theta$$ M1A1

use double angle formula

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \pi \int_0^{\frac{\pi}{3}} (4\sin\theta\cos\theta)^2 \times \sec^2\theta \, d\theta$$

$\sec\theta = \frac{1}{\cos\theta}$

$$= \pi \int_0^{\frac{\pi}{3}} 16\sin^2\theta\cos^2\theta \times \frac{1}{\cos^2\theta} \, d\theta$$

dM1

$$= \pi \int_0^{\frac{\pi}{3}} 16\sin^2\theta \, d\theta$$ A1

use double angle formula

$\cos 2A = 1 - 2\sin^2 A$

$$= \pi \int_0^{\frac{\pi}{3}} 8(1 - \cos 2\theta) \, d\theta$$ dM1

$2\sin^2 A = 1 - \cos 2A$

$$V = \int_0^{\frac{\pi}{3}} 8\pi(1 - \cos 2\theta) \, d\theta$$

Hence shown

A1

Leave blank

Question 9 continued

(b) Integrate what we showed in (a):

$$V = \int_0^{\frac{\pi}{3}} 8\pi(1-\cos 2\theta)\,d\theta$$

$$= 8\pi\int_0^{\frac{\pi}{3}} 1-\cos 2\theta \;\, d\theta$$ take out constant

$$= 8\pi\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$ B1

$$= 8\pi\left(\left(\tfrac{\pi}{3}-\tfrac{1}{2}\sin\tfrac{2\pi}{3}\right)-\cancel{\left(0-\tfrac{1}{2}\sin 0\right)}^{\,0}\right)$$ Sub. in limits

$$= 8\pi\left(\tfrac{\pi}{3}-\tfrac{1}{2}\times\tfrac{\sqrt{3}}{2}\right)$$

$$= \tfrac{8}{3}\pi^2 - 2\sqrt{3}\pi$$ M1A1

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

P 6 7 7 5 4 A 0 2 2 2 8

Leave blank

Question 9 continued

Q9

(Total 10 marks)

P 6 7 7 5 4 A 0 2 3 2 8

Leave blank

10. (a) Write $\dfrac{1}{(H-5)(H+3)}$ in partial fraction form. **(3)**

The depth of water in a storage tank is being monitored.

The depth of water in the tank, H metres, is modelled by the differential equation

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{(H-5)(H+3)}{40}$$

where t is the time, in days, from when monitoring began.

Given that the initial depth of water in the tank was 13 m,

$t = 0,\ H = 13$

(b) solve the differential equation to show that

$$H = \frac{10 + 3\mathrm{e}^{-0.2t}}{2 - \mathrm{e}^{-0.2t}}$$

(7)

(c) Hence find the time taken for the depth of water in the tank to fall to 8 m.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

According to the model, the depth of water in the tank will eventually fall to k metres.

(d) State the value of the constant k.

(1)

(a) We are asked to do Partial Fractions

$$\frac{1}{(H-5)(H+3)} \equiv \frac{A}{(H-5)} + \frac{B}{(H+3)}$$

Manipulate this to get common denominator

$$\frac{1}{(H-5)(H+3)} = \frac{A(H+3)+B(H-5)}{(H-5)(H+3)}$$

Equate the numerators and substitute in values in order to get A, B and C.

$$1 = A(H+3)+B(H-5)$$

choose values to substitute in so that brackets become 0, to make your life easier!

$H=-3 \Rightarrow 1 = A\cancel{(0)} + B(-8)$ 0

$B = -\frac{1}{8}$

$H=5 \Rightarrow 1 = A(8) + B\cancel{(0)}$ 0

$A = \frac{1}{8}$ M1

$$\therefore \frac{1}{(H-5)(H+3)} = \frac{1}{8(H-5)} - \frac{1}{8(H+3)}$$ A1

Leave blank

Question 10 continued

(b) $$\frac{dH}{dt} = -\frac{(H-5)(H+3)}{40}$$

Separate Variables by grouping the like terms on one side

$$\int \frac{1}{(H-5)(H+3)}\,dH = \int -\frac{1}{40}\,dt$$ M1

Substitute the P.F Result from (a) on RHS. $$\int \frac{1}{8(H-5)} - \frac{1}{8(H+3)}\,dH = -\frac{1}{40}t + c$$ Integrate RHS

$$\frac{1}{8}\ln|H-5| - \frac{1}{8}\ln|H+3| = -\frac{1}{40}t + c$$ M1A1

Use the given $t=0, H=13$ to get the value of c:

$$\frac{1}{8}\ln(13-5) - \frac{1}{8}\ln(13+3) = -\frac{1}{40}(0)+c$$

$$\frac{1}{8}\ln(8) - \frac{1}{8}\ln(16) = c$$

$$\frac{1}{8}\ln\frac{8}{16} = c \Rightarrow \frac{1}{8}\ln\frac{1}{2} = c$$ apply ln law $\ln a - \ln b = \ln\frac{a}{b}$

$$c = \frac{1}{8}\ln(2)^{-1} = -\frac{1}{8}\ln 2 = c$$ apply ln law $\ln a^b = b\ln a$

M1

Hence: $$\frac{1}{8}\ln|H-5| - \frac{1}{8}\ln|H+3| = -\frac{1}{40}t - \frac{1}{8}\ln 2$$ A1

$$\frac{1}{8}\ln\left(\frac{H-5}{H+3}\right) + \frac{1}{8}\ln 2 = -\frac{1}{40}t$$ apply ln law $\ln a - \ln b = \ln\frac{a}{b}$

$$\frac{1}{8}\ln\left(2\times\frac{H-5}{H+3}\right) = -\frac{1}{40}t$$ apply ln law $\ln a + \ln b = \ln a{\times}b$

$$\ln\left(\frac{2(H-5)}{H+3}\right) = -\frac{1}{5}t$$ multiply both sides by 8

$$e^{\ln\left(\frac{2(H-5)}{H+3}\right)} = e^{-0.2t}$$

$$2\times\frac{H-5}{H+3} = e^{-0.2t}$$ apply $e^{\ln x} = x$

$$H-5 = \frac{1}{2}e^{-0.2t}(H+3)$$ cross multiply the $(H+3)$, divide by 2 both sides

$$H-5 = \frac{1}{2}He^{-0.2t} + \frac{3}{2}e^{-0.2t}$$ expand RHS

$$H - \frac{1}{2}He^{-0.2t} = \frac{3}{2}e^{-0.2t} + 5$$ Rearrange to get all H's on one side

P 6 7 7 5 4 A 0 2 5 2 8

Leave blank

Question 10 continued

(b) cont.

$$H - \frac{1}{2}He^{-0.2t} = \frac{3}{2}e^{-0.2t} + 5$$

$$H\left(1 - \frac{1}{2}e^{-0.2t}\right) = \frac{3}{2}e^{-0.2t} + 5$$

$$H = \frac{\frac{3}{2}e^{-0.2t} + 5}{1 - \frac{1}{2}e^{-0.2t}}$$ dddM1

multiply up and down by 2 to get rid of the fractions

$$\boxed{H = \frac{3e^{-0.2t} + 10}{2 - e^{-0.2t}}}$$

hence shown

A1

(c) Substitute $H = 8$ into the solved D.E.

$$8 = \frac{3e^{-0.2t} + 10}{2 - e^{-0.2t}}$$ Solve for t

$$16 - 8e^{-0.2t} = 3e^{-0.2t} + 10$$ Cross multiply $(2 - e^{-0.2t})$

$$6 = 11e^{-0.2t}$$ collect like terms

$$\frac{6}{11} = e^{-0.2t}$$ M1

$$\ln\frac{6}{11} = \ln e^{-0.2t}$$ use $\ln e^{x} = x$

$$\ln\frac{6}{11} = -0.2t$$

$$t = -5\ln\frac{6}{11}$$

$$\boxed{\therefore t = 3.03 \text{ days}}$$ A1

(d) As $t \to \infty$:

$$H = \frac{3\cancel{e^{-0.2t}}^{\,0} + 10}{2 - \cancel{e^{-0.2t}}^{\,0}}$$

$$H = \frac{10}{2} = 5\text{m}.$$

$$\boxed{k = 5}$$ B1

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Leave blank

Question 10 continued

Leave blank

Question 10 continued

Q10

(Total 14 marks)

TOTAL FOR PAPER IS 75 MARKS

END